Journal of Intelligent Decision and Computational Modelling

www.jidcm.reapress.com

J. Intell. Decis. Comput. Model.Vol. 2, No. 3 (2026) 175–190.

Paper Type: Original Article

On De-Roughication, De-Hyperization, and De-Nearization: Reduction Principles in Advanced Structural System

Takaaki Fujita

Independent Researcher, Tokyo, Japan; Takaaki.fujita060@gmail.com.

Citation:

Received: 14 January 2026 Revised: 21 March 2026 Accepted: 02 June 2026	Fujita, T. (2026). On de-roughication, de-hyperization, and de nearization: Reduction principles in advanced structural systems. *Journal of intelligent decision and computational modelling, 2*(3), 175-190.

Abstract

This paper formalizes three "de-" processes that convert higher-order uncertainty into crisp structures. First, we define a cost-sensitive deroughication operator $D_{\alpha,\beta}$ that selects boundary elements by rough-membership thresholds; we prove sandwich, extremal, idempotence, and monotonicity properties and show it minimizes a risk. Second, we specify de-hyperization from (m,n)–SuperHyperStructures to HyperStructures and then to classical operations via lifting/flattening and a canonical selector, with a recovery result for lifted classical operations. Third, we introduce de-nearization $\mathrm{DeNear}_{\varepsilon}$ that removes conflicting descriptions to enforce an ε margin; we prove termination, separation, and maximality. Applications include medical triage, fraud screening, manufacturing, and transit.

Keywords: De-roughication, De-hyperization, De-nearization, Rough set, HyperStructure, SuperHyperStructure, Near set.

Corresponding Author: Takaaki.fujita060@gmail.com

https://doi.org/10.48314/jidcm.vi.77

Licensee System Analytics. This article is an open-access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0).

1|Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Throughout this paper, all structures and sets are assumed to be finite.

1.1|Rough Set

A rough set approximates a subset by means of lower and upper approximations defined through equivalence classes, thereby providing a systematic framework for representing certainty, uncertainty, and boundary information in set membership [1, 2]. Numerous extensions of rough-set theory have also been developed, including fuzzy rough sets [3, 4, 5], intuitionistic fuzzy rough sets [6, 7], neutrosophic rough sets [8, 9, 10], soft rough sets [11, 12], uncertain rough sets [13], and plithogenic rough sets [14, 15, 16]. These developments demonstrate the broad applicability and continuing importance of rough-set research in the analysis of incomplete, imprecise, and uncertain information. Furthermore, related concepts such as rough graphs [17, 18] and rough algebras [19] are also known. The fundamental definitions are provided below.

Definition 1. (Rough Set Approximation)**.** [20] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

(1) *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

(2) *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 1. (Concrete computation of lower/upper approximations)**.** Let the universe and indiscernibility (equivalence) classes be

$$X = \{1, 2, 3, 4, 5, 6, 7, 8\}, \qquad \begin{aligned} [1]_R = [2]_R &= \{1, 2\} =: C_1, \\ [3]_R = [4]_R = [5]_R &= \{3, 4, 5\} =: C_2, \\ [6]_R = [7]_R &= \{6, 7\} =: C_3, \\ [8]_R &= \{8\} =: C_4. \end{aligned}$$

Fix the target set $U = \{1, 2, 6, 8\}$.

Lower approximation $\underline{U}$**.** We include those $x \in X$ whose entire class is inside U:

$$\begin{aligned} C_1 = \{1, 2\} \subseteq U &\;\Rightarrow\; \{1, 2\} \subseteq \underline{U}, \\ C_2 = \{3, 4, 5\} \not\subseteq U &\;\Rightarrow\; \{3, 4, 5\} \cap \underline{U} = \emptyset, \\ C_3 = \{6, 7\} \not\subseteq U &\;\Rightarrow\; \{6, 7\} \cap \underline{U} = \emptyset, \\ C_4 = \{8\} \subseteq U &\;\Rightarrow\; \{8\} \subseteq \underline{U}. \end{aligned}$$

Hence

$$\underline{U} = \{1, 2, 8\}.$$

Upper approximation $\overline{U}$. We include those $x \in X$ whose class meets U:

$$C_1 \cap U = \{1,2\} \neq \emptyset \Rightarrow \{1,2\} \subseteq \overline{U},$$
$$C_2 \cap U = \emptyset \Rightarrow \{3,4,5\} \cap \overline{U} = \emptyset,$$
$$C_3 \cap U = \{6\} \neq \emptyset \Rightarrow \{6,7\} \subseteq \overline{U},$$
$$C_4 \cap U = \{8\} \neq \emptyset \Rightarrow \{8\} \subseteq \overline{U}.$$

Thus

$$\overline{U} = \{1,2,6,7,8\}.$$

Verification and regions. We have

$$\underline{U} = \{1,2,8\} \ \subseteq \ U = \{1,2,6,8\} \ \subseteq \ \overline{U} = \{1,2,6,7,8\}.$$

Boundary and negative regions are

$$\mathrm{BND}_R(U) = \overline{U} \setminus \underline{U} = \{6,7\}, \qquad \mathrm{NEG}_R(U) = X \setminus \overline{U} = \{3,4,5\}.$$

Rough membership (constant on classes). For $x \in X$,

$$\mu_U(x) = \frac{|[x]_R \cap U|}{|[x]_R|} = \begin{cases} 1 & (x \in C_1 = \{1,2\}), \\ 0 & (x \in C_2 = \{3,4,5\}), \\ \frac{1}{2} & (x \in C_3 = \{6,7\}), \\ 1 & (x \in C_4 = \{8\}). \end{cases}$$

Hence elements in $\underline{U}$ have $\mu_U = 1$, elements outside $\overline{U}$ have $\mu_U = 0$, and the boundary $\{6,7\}$ carries uncertainty $\mu_U = \frac{1}{2}$.

1.2|Classical Structure, Hyperstructure, and n-Superhyperstructure

A *Classical Structure* is a broad mathematical framework. A *Hyperstructure* is obtained by applying the powerset construction [21, 22, 23], while an *n-Superhyperstructure* arises by iterating that construction n times [24, 25, 26, 27]. Intuitively, the n-th powerset is the result of repeatedly applying the powerset operator n times. The basic definitions and concise examples are recalled below.

Definition 2. (Powerset)**.** [28] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

Definition 3. (n^{th} Powerset)**.** (cf. [24]) For any set H, define the iterated powersets by

$$\mathcal{P}^1(H) = \mathcal{P}(H), \qquad \mathcal{P}^{n+1}(H) = \mathcal{P}(\mathcal{P}^n(H)) \quad (n \geq 1).$$

Equivalently, set $\mathcal{P}^0(H) = H$ and for all $k \geq 0$,

$$\mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)).$$

The *nonempty* n^{th} powerset removes the empty set at each stage. Writing $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$, define

$$\mathcal{P}^{*1}(H) = \mathcal{P}^*(H), \qquad \mathcal{P}^{*(n+1)}(H) = \mathcal{P}^*(\mathcal{P}^{*n}(H)) \quad (n \geq 1).$$

Example 2. (University course planning as an iterated powerset model)**.** Let the base set of available single–semester courses be

$$H = \{\text{Math(M)},\ \text{CS(C)},\ \text{Physics(P)}\}.$$

Then $|H| = 3$ and

$$\mathcal{P}(H) = \{\emptyset, \{M\}, \{C\}, \{P\}, \{M,C\}, \{M,P\}, \{C,P\}, \{M,C,P\}\}, \quad |\mathcal{P}(H)| = 2^3 = 8.$$

Level 1 (study plans): an individual *study plan* is any subset of courses, e.g.

$$S_1 = \{M, C\}, \quad S_2 = \{C, P\}, \quad S_3 = \{M, P\} \ \in \ \mathcal{P}(H).$$

Level 2 (degree templates): a *degree template* is a set of admissible study plans; hence it is an element of

$$\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H)).$$

For instance,

$$T = \{\, S_1,\ S_2 \,\} = \{\, \{M, C\},\ \{C, P\} \,\} \ \in \ \mathcal{P}^2(H),$$

meaning "either take {M,C} or {C,P}."

Level 3 (catalog options): a *catalog option* is a set of degree templates, i.e. an element of

$$\mathcal{P}^3(H) = \mathcal{P}(\mathcal{P}^2(H)).$$

For example,

$$\mathcal{K} = \{\, T,\ T' \,\} \quad \text{with} \quad T' = \{\, \{M, C, P\},\ \{C\} \,\},$$

so $\mathcal{K} \in \mathcal{P}^3(H)$ is interpreted as a *catalog* offering two alternative degree templates T and T'.

Membership checks (explicit):

$$S_1 \in \mathcal{P}(H), \qquad T \in \mathcal{P}(\mathcal{P}(H)) = \mathcal{P}^2(H), \qquad \mathcal{K} \in \mathcal{P}(\mathcal{P}^2(H)) = \mathcal{P}^3(H).$$

If empty choices are disallowed, replace each $\mathcal{P}$ by the nonempty variant $\mathcal{P}^*$; then $S_i \in \mathcal{P}^*(H)$, $T \in \mathcal{P}^{*2}(H)$, and $\mathcal{K} \in \mathcal{P}^{*3}(H)$.

Definition 4. (Classical Structure)**.** (cf. [24, 29]) A *Classical Structure* is a mathematical system on a nonempty set H, equipped with one or more *Classical Operations* satisfying designated *Classical Axioms*. Concretely, a *Classical Operation* is a map

$$\#_0 : H^m \to H,$$

where $m \geq 1$ and H^m is the m-fold Cartesian product of H. Typical examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 5. (Hyperoperation)**.** (cf. [30, 31]) A *hyperoperation* generalizes a binary operation by allowing the result of combining two inputs to be a set rather than a single element. For a set S, a hyperoperation $\circ$ is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Definition 6. (Hyperstructure)**.** (cf. [24, 32, 33, 34, 29]) A *Hyperstructure* extends a Classical Structure by operating on the powerset of a base set. Formally,

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ its powerset, and $\circ$ an operation defined on subsets of S. Hyperstructures thus enable operations on collections of elements, not only on single elements.

Example 3. (HyperStructure in public transit: two–leg layover selector)**.** Let $S = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$ be bus stops in a small network. Directed links (one–hop rides) are

$$\mathrm{A} \to \mathrm{C},\ \mathrm{A} \to \mathrm{D},\ \mathrm{B} \to \mathrm{C},\ \mathrm{C} \to \mathrm{D},\ \mathrm{C} \to \mathrm{E},\ \mathrm{D} \to \mathrm{E}.$$

Define a hyperstructure $(\mathcal{P}(S), \circ)$ by the operation

$$\circ(A, B) \ := \ \{\, v \in S \ \mid \ \exists a \in A,\ \exists b \in B : \ (a \to v) \ \wedge \ (v \to b) \,\}, \qquad A, B \subseteq S.$$

Thus $\circ(A, B)$ returns the set of *layover stops* v enabling a 2-leg route from some $a \in A$ to some $b \in B$.

Concrete computation with $A = \{\mathrm{A}, \mathrm{B}\}$ and $B = \{\mathrm{D}, \mathrm{E}\}$:

$$\text{From A} : \mathrm{A} \to \{\mathrm{C}, \mathrm{D}\};\ \mathrm{C} \to \{\mathrm{D}, \mathrm{E}\}\ (\Rightarrow \mathrm{C} \in \circ),\ \mathrm{D} \to \{\mathrm{E}\}\ (\Rightarrow \mathrm{D} \in \circ).$$
$$\text{From B} : \mathrm{B} \to \{\mathrm{C}\};\ \mathrm{C} \to \{\mathrm{D}, \mathrm{E}\}\ (\Rightarrow \mathrm{C} \in \circ).$$

Therefore

$$\circ(\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{D}, \mathrm{E}\}) = \{\mathrm{C}, \mathrm{D}\} \subseteq S.$$

This is a real–world hyperstructure: the "product'' of two sets of origins/destinations is a *set* of feasible transfer points rather than a single element.

Definition 7. (SuperHyperOperations)**.** (cf. [24]) Let H be nonempty and write $\mathcal{P}^0(H) = H$, $\mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H))$ for $k \geq 0$. A *SuperHyperOperation* of order (m, n) is an m-ary mapping

$$\circ^{(m,n)} : H^m \longrightarrow \mathcal{P}_*^n(H),$$

where $\mathcal{P}_*^n(H)$ denotes either the n-th powerset with the empty set excluded (classical-type) or included (neutrosophic-type):

- If the codomain excludes $\emptyset$, we speak of a *classical-type* (m, n)*-SuperHyperOperation*.
- If the codomain allows $\emptyset$, we speak of a *neutrosophic* (m, n)*-SuperHyperOperation*.

These maps generalize hyperoperations to multi-level settings via iterated powersets.

Definition 8. (n-Superhyperstructure)**.** (cf. [24, 29]) An n*-Superhyperstructure* further generalizes a Hyperstructure by operating on the n-th powerset:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ the n-th powerset of S, and $\circ$ an operation defined on $\mathcal{P}_n(S)$. This iterative framework supports increasingly hierarchical and complex relationships.

Definition 9. (SuperHyperStructure of order (m, n))**.** (cf. [35]) Let S be nonempty and $m, n \geq 0$. A (m, n)*-SuperHyperStructure* of arity s is any map

$$\odot^{(m,n)} : \left(\mathcal{P}^m(S)\right)^s \longrightarrow \mathcal{P}^n(S).$$

Special cases recover standard settings: $m = n = 0$ yields ordinary s-ary operations; $m = 0, n = 1$ yields hyperoperations; and $s = 1$ corresponds to *superhyperoperations*.

Example 4. (SuperHyperStructure of order $(m, n) = (1, 2)$ for meal planning bundles)**.** Let $S = \{\mathrm{Ch}, \mathrm{T}, \mathrm{Br}, \mathrm{Sp}, \mathrm{R}, \mathrm{Q}\}$ denote foods: chicken (Ch), tofu (T), broccoli (Br), spinach (Sp), rice (R), quinoa (Q). We form 3–item dinner bundles: one protein, one vegetable, one starch.

Define the $(m, n) = (1, 2)$ superhyperoperation (arity $s = 3$)

$$\odot^{(1,2)} : \mathcal{P}(S)^3 \longrightarrow \mathcal{P}^2(S), \qquad \odot^{(1,2)}(A, B, C) := \left\{ Y \subseteq S \;\middle|\; \begin{array}{l} |Y| = 3, \\ |Y \cap A| = 1,\ |Y \cap B| = 1,\ |Y \cap C| = 1 \end{array} \right\},$$

i.e., the output is a *family of menus* (a set of 3–element subsets), hence an element of $\mathcal{P}^2(S)$.

Take concrete ingredient sets:

$$A = \{\mathrm{Ch}, \mathrm{T}\}\ (\text{proteins}), \quad B = \{\mathrm{Br}, \mathrm{Sp}\}\ (\text{vegetables}), \quad C = \{\mathrm{R}\}\ (\text{starches}).$$

Then the superhyperoutput enumerates all feasible menus (exactly $|A| \cdot |B| \cdot |C| = 2 \cdot 2 \cdot 1 = 4$):

$$\begin{aligned} \odot^{(1,2)}(A, B, C) = \{&\{\mathrm{Ch}, \mathrm{Br}, \mathrm{R}\},\ \{\mathrm{Ch}, \mathrm{Sp}, \mathrm{R}\},\ \{\mathrm{T}, \mathrm{Br}, \mathrm{R}\},\ \{\mathrm{T}, \mathrm{Sp}, \mathrm{R}\}\} \\ &\in \mathcal{P}^2(S). \end{aligned}$$

Real–world reading: inputs are *sets of available items* per category, and the output is a *set of menus* (higher–level objects), which is precisely a $(1, 2)$–SuperHyperStructure behavior.

1.3|Near Set

A near set consists of subsets whose feature-space images intersect, revealing their elements share sufficiently similar descriptive attributes or qualities (cf.[36, 37, 38, 39]).

Definition 10. (Near Set)**.** Let X be a nonempty set and let

$$\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$$

be a finite family of *probe functions* $\phi_i : X \to \mathbb{R}$. Define the *feature map*

$$\Phi : X \longrightarrow \mathbb{R}^n, \qquad \Phi(x) = (\phi_1(x), \dots, \phi_n(x)),$$

and extend it to subsets by

$$\mathcal{Q}(A) \;=\; \big\{\Phi(a) \;\big|\; a \in A\big\} \;\subseteq\; \mathbb{R}^n, \quad A \subseteq X.$$

Endow $\mathbb{R}^n$ with its usual Euclidean topology, and write $\overline{S}$ for the closure of $S \subseteq \mathbb{R}^n$. Then the *descriptive proximity* relation δ_Φ on 2^X is defined by

$$A \;\delta_\Phi\; B \quad\Longleftrightarrow\quad \overline{\mathcal{Q}(A)} \;\cap\; \overline{\mathcal{Q}(B)} \;\neq\; \emptyset,$$

for any $A, B \subseteq X$. Two subsets A, B satisfying $A\delta_\Phi B$ are called *descriptively near sets.* In the special case when each ϕ_i is the identity on a metric space (X, d), this reduces to the spatial nearness condition

$$A \;\delta\; B \quad\Longleftrightarrow\quad A \cap B \neq \emptyset.$$

Example 5. (Image patches with photometric–textural probes: $A \;\delta_\Phi\; B$ holds)**.** Let $X = \{p_1, \dots, p_6\}$ denote 16×16 grayscale image patches. Define two probe functions (photometric mean and edge density)

$$\phi_1(x) = \text{mean intensity in } [0, 255], \qquad \phi_2(x) = \text{Canny edge pixel ratio in } [0, 1].$$

Thus $\Phi(x) = (\phi_1(x), \phi_2(x)) \in \mathbb{R}^2$. Suppose the measured features are

$$\Phi(p_1) = (120,\, 0.20), \quad \Phi(p_2) = (118,\, 0.21), \quad \Phi(p_3) = (100,\, 0.05),$$
$$\Phi(p_4) = (100,\, 0.05), \quad \Phi(p_5) = (60,\, 0.04), \quad \Phi(p_6) = (200,\, 0.30).$$

Take the subsets $A = \{p_1, p_2, p_3\}$ and $B = \{p_4, p_5\}$. Then

$$\mathcal{Q}(A) = \{(120, 0.20), (118, 0.21), (100, 0.05)\}, \qquad \mathcal{Q}(B) = \{(100, 0.05), (60, 0.04)\}.$$

Both $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are finite, hence closed in $\mathbb{R}^2$: $\overline{\mathcal{Q}(A)} = \mathcal{Q}(A)$ and $\overline{\mathcal{Q}(B)} = \mathcal{Q}(B)$. Their intersection is nonempty because

$$(100, 0.05) \in \mathcal{Q}(A) \cap \mathcal{Q}(B).$$

Therefore,

$$\overline{\mathcal{Q}(A)} \cap \overline{\mathcal{Q}(B)} \neq \emptyset \quad\Longrightarrow\quad A \;\delta_\Phi\; B.$$

Interpretation: although A and B contain different patches, they are *descriptively near* since both contain a patch with the same photometric–textural description $(100, 0.05)$.

Example 6. (Image patches with photometric–textural probes: $A \;\delta_\Phi\; B$ holds)**.** Let $X = \{p_1, \dots, p_6\}$ denote 16×16 grayscale image patches. Define two probe functions (photometric mean and edge density)

$$\phi_1(x) = \text{mean intensity in } [0, 255], \qquad \phi_2(x) = \text{Canny edge pixel ratio in } [0, 1].$$

Thus $\Phi(x) = (\phi_1(x), \phi_2(x)) \in \mathbb{R}^2$. Suppose the measured features are

$$\Phi(p_1) = (120,\, 0.20), \quad \Phi(p_2) = (118,\, 0.21), \quad \Phi(p_3) = (100,\, 0.05),$$
$$\Phi(p_4) = (100,\, 0.05), \quad \Phi(p_5) = (60,\, 0.04), \quad \Phi(p_6) = (200,\, 0.30).$$

Take the subsets $A = \{p_1, p_2, p_3\}$ and $B = \{p_4, p_5\}$. Then

$$\mathcal{Q}(A) = \{(120, 0.20), (118, 0.21), (100, 0.05)\}, \qquad \mathcal{Q}(B) = \{(100, 0.05), (60, 0.04)\}.$$

Both $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are finite, hence closed in $\mathbb{R}^2$: $\overline{\mathcal{Q}(A)} = \mathcal{Q}(A)$ and $\overline{\mathcal{Q}(B)} = \mathcal{Q}(B)$. Their intersection is nonempty because

$$(100, 0.05) \in \mathcal{Q}(A) \cap \mathcal{Q}(B).$$

Therefore,

$$\overline{\mathcal{Q}(A)} \cap \overline{\mathcal{Q}(B)} \neq \emptyset \quad\Longrightarrow\quad A \;\delta_\Phi\; B.$$

Interpretation: although A and B contain different patches, they are *descriptively near* since both contain a patch with the same photometric–textural description $(100, 0.05)$.

Example 7. (Consumer devices with price–weight probes: $A\ \delta_\Phi\ B$ holds)**.** Let $X = \{\ell_1, \ell_2, \ell_3, t_1, t_2, t_3\}$ be devices: three laptops (ℓ_i) and three tablets (t_j). Define probes

$$\phi_1(x) = \frac{\text{price}(x)}{100} \in \mathbb{R}, \qquad \phi_2(x) = \text{portability index} = \begin{cases} 1, & \text{weight}(x) \le 1.30 \text{ kg}, \\ 2, & \text{weight}(x) > 1.30 \text{ kg}. \end{cases}$$

Thus $\Phi(x) = (\phi_1(x), \phi_2(x)) \in \mathbb{R}^2$. Suppose

$$\Phi(\ell_1) = (12, 1), \quad \Phi(\ell_2) = (10, 2), \quad \Phi(\ell_3) = (9, 1),$$
$$\Phi(t_1) = (6, 1), \quad \Phi(t_2) = (10, 1), \quad \Phi(t_3) = (12, 1).$$

Take the subsets $A = \{\ell_1, \ell_2\}$ (two laptops) and $B = \{t_2, t_3\}$ (two tablets). Then

$$\mathcal{Q}(A) = \{(12, 1), (10, 2)\}, \qquad \mathcal{Q}(B) = \{(10, 1), (12, 1)\}.$$

Again $\mathcal{Q}(A), \mathcal{Q}(B)$ are finite, hence $\overline{\mathcal{Q}(A)} = \mathcal{Q}(A)$ and $\overline{\mathcal{Q}(B)} = \mathcal{Q}(B)$. We have

$$(12, 1) \in \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset,$$

so

$$A\ \delta_\Phi\ B.$$

Interpretation: despite A (laptops) and B (tablets) being disjoint in object space, they are *descriptively near* because both contain a device with the same feature description "\$1200 price band and light weight'' $(12, 1)$.

2|Main Results: Deroughication

Intuitively, deroughication selects, for each boundary element $x \in \overline{U} \setminus \underline{U}$, whether to include x in a crisp output set B by comparing $\mu_U(x)$ against a cost-induced threshold.

Definition 11. (Deroughication Operator $D_{\alpha,\beta}$)**.** Fix misclassification costs $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$ where

$$\alpha = \text{cost of a false positive (include } x \notin U), \quad \beta = \text{cost of a false negative (exclude } x \in U).$$

Define the threshold

$$\tau(\alpha, \beta) := \frac{\alpha}{\alpha + \beta} \in [0, 1].$$

Given (X, R, U), the *deroughication* $D_{\alpha,\beta}(U)$ is the crisp set

$$D_{\alpha,\beta}(U) \ := \ \underline{U} \ \cup \ \{\, x \in \overline{U} \setminus \underline{U} \ \mid \ \mu_U(x) \ge \tau(\alpha, \beta) \,\}.$$

For the extremal case $\alpha = 0$ (no false-positive penalty), we *define* $D_{0,\beta}(U) := \overline{U}$; for the limiting case $\alpha \to \infty$ (infinite false-positive penalty), we have $D_{\alpha,\beta}(U) \to \underline{U}$.

The above decision rule arises from minimizing expected cost per element on the boundary. If one predicts "in U" for x, the expected cost is $\alpha \cdot (1 - \mu_U(x))$; if one predicts "not in U", the expected cost is $\beta \cdot \mu_U(x)$. Including x is optimal iff

$$\alpha(1 - \mu_U(x)) \ \le \ \beta\mu_U(x) \quad \Longleftrightarrow \quad \mu_U(x) \ \ge \ \frac{\alpha}{\alpha + \beta} \ = \ \tau(\alpha, \beta).$$

Proposition 1. (Sandwich, Extremes, Idempotence, Monotonicity)**.** *For any (X, R, U) and $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$:*

(i) ***Sandwich:*** $\underline{U} \ \subseteq \ D_{\alpha,\beta}(U) \ \subseteq \ \overline{U}$.

(ii) ***Extremes:*** $D_{0,\beta}(U) = \overline{U}$ *and* $\lim_{\alpha\to\infty} D_{\alpha,\beta}(U) = \underline{U}$.

(iii) ***Idempotence on definable sets:*** *If U is R-definable ($\underline{U} = \overline{U} = U$), then $D_{\alpha,\beta}(U) = U$ for all α, β.*

(iv) ***Monotonicity in the threshold:*** *If $\tau_1 \le \tau_2$, then $D_{\alpha_2,\beta_2}(U) \subseteq D_{\alpha_1,\beta_1}(U)$ where $\tau_i = \tau(\alpha_i, \beta_i)$.*

Proof: (i) By Definition 2.1, we include $\underline{U}$ and only select additional elements from the boundary $\overline{U} \setminus \underline{U}$, hence the inclusion chain holds.

(ii) If $\alpha = 0$, then $\tau = 0$ and every boundary element satisfies $\mu_U(x) \geq 0$, so all boundary points are included, yielding $\overline{U}$. As $\alpha \to \infty$, $\tau(\alpha, \beta) = \alpha/(\alpha + \beta) \to 1$, so no boundary element with $\mu_U(x) < 1$ is included, and the limit is $\underline{U}$.

(iii) If U is definable, then for all $x \in U$ we have $\mu_U(x) = 1$ and for all $x \notin U$ we have $\mu_U(x) = 0$. Thus $D_{\alpha,\beta}(U) = U \cup \{x \in U \setminus U : \cdots\} = U$.

(iv) If $\tau_1 \leq \tau_2$, then $\{x : \mu_U(x) \geq \tau_2\} \subseteq \{x : \mu_U(x) \geq \tau_1\}$, hence the asserted set inclusion follows from the definition. □

Remark 1. (Equivalence to Cost Minimization over Crisp Selections)**.** Let $\mathcal{F} := \{B \mid \underline{U} \subseteq B \subseteq U\overline{\}}$. Consider the separable risk

$$\mathcal{R}_{\alpha,\beta}(B) \;:=\; \sum_{x \in B \setminus \underline{U}} \alpha \cdot (1 - \mu_U(x)) \;+\; \sum_{x \in \overline{U} \setminus B} \beta \cdot \mu_U(x), \quad B \in \mathcal{F}.$$

By elementwise comparison, $D_{\alpha,\beta}(U)$ minimizes $\mathcal{R}_{\alpha,\beta}$ over $\mathcal{F}$. This minimizer is unique provided that

$$\mu_U(x) \neq \tau(\alpha, \beta)$$

for every $x \in \overline{U} \setminus \underline{U}$. If equality holds for some boundary element, both its inclusion and exclusion yield the same minimum cost.

Example 8. (Medical triage with cost–sensitive deroughication)**.** Consider $X = \{1, 2, \ldots, 12\}$ patients. The indiscernibility R groups patients by (age band, CRP band) into

$$C_1 = \{1, 2\}, \quad C_2 = \{3, 4, 5\}, \quad C_3 = \{6, 7, 8\}, \quad C_4 = \{9, 10, 11, 12\}.$$

Let $U \subseteq X$ be the set of *truly infected* patients (confirmed later by gold standard):

$$U = \{1, 2, 3, 4, 6, 9\}.$$

Lower/upper approximations:

$$\underline{U} = \{1, 2\} \quad (\text{since } C_1 \subseteq U), \qquad \overline{U} = X \quad (\text{each } C_i \text{ meets } U).$$

Rough membership per class (constant on each C_i):

$$\mu_U(x) = \frac{|[x]_R \cap U|}{|[x]_R|} = \begin{cases} 1 & (x \in C_1), \\ \frac{2}{3} \approx 0.666\ldots & (x \in C_2), \\ \frac{1}{3} \approx 0.333\ldots & (x \in C_3), \\ \frac{1}{4} = 0.25 & (x \in C_4). \end{cases}$$

Suppose a false negative (missed infection) costs $\beta = 2$ and a false positive (unnecessary CT) costs $\alpha = 1$. The deroughication threshold is

$$\tau = \frac{\alpha}{\alpha + \beta} = \frac{1}{1 + 2} = \frac{1}{3} \approx 0.333\ldots.$$

By Definition $D_{\alpha,\beta}(U) = \underline{U} \cup \{x \in \overline{U} \setminus \underline{U} : \mu_U(x) \geq \tau\}$, we decide per class:

$\mu_U(C_2) = \frac{2}{3} \geq \frac{1}{3} \Rightarrow C_2 \subset D_{1,2}(U)$, $\mu_U(C_3) = \frac{1}{3} \geq \frac{1}{3} \Rightarrow C_3 \subset D_{1,2}(U)$, $\mu_U(C_4) = 0.25 < \frac{1}{3} \Rightarrow C_4 \cap D_{1,2}(U) = \emptyset$.

Hence

$$D_{1,2}(U) = \{1, 2\} \cup \{3, 4, 5\} \cup \{6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Elementwise check (explicit inequalities):

$$\mu_U(5) = \tfrac{2}{3} \geq \tfrac{1}{3} \Rightarrow 5 \in D_{1,2}(U), \quad \mu_U(7) = \tfrac{1}{3} \geq \tfrac{1}{3} \Rightarrow 7 \in D_{1,2}(U), \quad \mu_U(10) = 0.25 < \tfrac{1}{3} \Rightarrow 10 \notin D_{1,2}(U).$$

Sandwich holds: $\underline{U} = \{1, 2\} \subset D_{1,2}(U) \subset \overline{U} = X$. Operationally, the hospital escalates patients in $D_{1,2}(U)$ for confirmatory imaging; C_4 receives watchful waiting.

Example 9. (Credit-card fraud screening with asymmetric costs)**.** Let $X = \{t_1, \dots, t_{15}\}$ be transactions. The relation R groups them by behavior patterns:

$$C_1 = \{t_1, t_2, t_3, t_4\}, \quad C_2 = \{t_5, t_6, t_7, t_8, t_9\}, \quad C_3 = \{t_{10}, t_{11}, t_{12}\}, \quad C_4 = \{t_{13}, t_{14}, t_{15}\}.$$

Let U be *confirmed frauds* after investigation:

$$U = \{t_1, t_2, t_3, t_6, t_{13}\}.$$

Compute lower/upper approximations:

$$\underline{U} = \emptyset \quad \text{(no class fully in } U\text{)}, \qquad \overline{U} = C_1 \cup C_2 \cup C_4 \quad \text{(since } C_3 \cap U = \emptyset\text{)}.$$

Rough membership by class:

$$\mu_U(\cdot) = \begin{cases} \frac{3}{4} = 0.75 & (C_1), \\ \frac{1}{5} = 0.2 & (C_2), \\ 0 & (C_3), \\ \frac{1}{3} \approx 0.333\dots & (C_4). \end{cases}$$

Set costs: false positive (blocking a legit transaction) $\alpha = 3$; false negative (missing a fraud) $\beta = 7$. Then

$$\tau = \frac{\alpha}{\alpha + \beta} = \frac{3}{3 + 7} = 0.3.$$

Deroughication:

$$\mu_U(C_1) = 0.75 \geq 0.3 \Rightarrow C_1 \subset D_{3,7}(U), \quad \mu_U(C_2) = 0.2 < 0.3 \Rightarrow C_2 \cap D_{3,7}(U) = \emptyset,$$
$$\mu_U(C_4) = \tfrac{1}{3} \geq 0.3 \Rightarrow C_4 \subset D_{3,7}(U), \quad C_3 \cap U = \emptyset \Rightarrow C_3 \cap D_{3,7}(U) = \emptyset.$$

Therefore

$$D_{3,7}(U) = C_1 \cup C_4 = \{t_1, t_2, t_3, t_4, t_{13}, t_{14}, t_{15}\}.$$

Elementwise verification:

$$\mu_U(t_4) = 0.75 \geq 0.3 \Rightarrow t_4 \in D_{3,7}(U), \qquad \mu_U(t_6) = 0.2 < 0.3 \Rightarrow t_6 \notin D_{3,7}(U).$$

Interpretation: only the high-fraud and new-merchant classes are *crisp-flagged*; medium-risk class C_2 is *not* auto-blocked because α/β makes customer friction relatively costly.

Example 10. (Manufacturing quality control: hold/rework decision)**.** Ten lots $X = \{b_1, \dots, b_{10}\}$ are produced. Indiscernibility R groups by (supplier, line):

$$C_1 = \{b_1, b_2\}, \quad C_2 = \{b_3, b_4, b_5\}, \quad C_3 = \{b_6, b_7\}, \quad C_4 = \{b_8, b_9, b_{10}\}.$$

NDT (non-destructive test) later reveals lots with latent defects:

$$U = \{b_1, b_2, b_3, b_7\}.$$

Then

$$\underline{U} = C_1 = \{b_1, b_2\} \quad \text{(entire } C_1 \subset U\text{)}, \qquad \overline{U} = C_1 \cup C_2 \cup C_3.$$

Rough membership:

$$\mu_U(\cdot) = \begin{cases} 1 & (C_1), \\ \frac{1}{3} \approx 0.333\dots & (C_2), \\ \frac{1}{2} = 0.5 & (C_3), \\ 0 & (C_4). \end{cases}$$

Business costs: scrapping/tearing down a good lot (false positive) costs $\alpha = 5$; shipping a defective lot and facing recall (false negative) costs $\beta = 20$. Thus

$$\tau = \frac{\alpha}{\alpha + \beta} = \frac{5}{25} = 0.2.$$

Decision by $D_{\alpha,\beta}$:

$$\mu_U(C_2) = \tfrac{1}{3} \geq 0.2 \Rightarrow C_2 \subset D_{5,20}(U), \qquad \mu_U(C_3) = 0.5 \geq 0.2 \Rightarrow C_3 \subset D_{5,20}(U).$$

Hence

$$D_{5,20}(U) = C_1 \cup C_2 \cup C_3 = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}.$$

Explicit checks:

$$\mu_U(b_4) = \tfrac{1}{3} \geq 0.2 \Rightarrow b_4 \in D_{5,20}(U), \qquad \mu_U(b_9) = 0 < 0.2 \Rightarrow b_9 \notin D_{5,20}(U).$$

Policy reading: lots in $D_{5,20}(U)$ are placed on *hold/rework*; C_4 proceeds as *pass*. If the plant becomes more averse to scrapping (e.g. $\alpha = 15,\ \beta = 20$ so $\tau = 15/35 \approx 0.4286$), then C_2 drops below threshold while C_3 stays above, giving

$$D_{15,20}(U) = C_1 \cup C_3 = \{b_1, b_2, b_6, b_7\},$$

which concretely illustrates the monotonicity with respect to τ.

3|De-Hyperization from SuperHyperStructures to Hyper and Classical Structures

De-Hyperization is the reduction from SuperHyperStructures into HyperStructures, then into classical structures, through flattening and canonical selection operations.

Definition 12. (Iterated powerset, lifting, and flattening)**.** Let S be a nonempty finite set. Define $P^0(S) := S$ and $P^{r+1}(S) := P(P^r(S))$ for $r \geq 0$.

(a) Nesting (lift) into higher levels. For any $r \geq 1$, define the canonical lift $\iota_r : P(S) \to P^r(S)$ recursively by

$$\iota_1(A) := A \quad (\subseteq S), \qquad \iota_{r+1}(A) := \{\, \iota_r(A) \,\} \in P^{r+1}(S).$$

Thus $\iota_r(A)$ is the r-fold nested singleton containing A.

(b) Level-wise flattening of outputs. For $q \geq 1$, define $\mathrm{Flat}_q : P^q(S) \to P(S)$ by

$$\mathrm{Flat}_1(B) := B\ (\subseteq S), \qquad \mathrm{Flat}_{q+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_q(Y).$$

By construction, for all $A \subseteq S$ and $q \geq 1$,

$$\mathrm{Flat}_q(\iota_q(A)) = A.$$

Definition 13. (Stage 1: de-hyperization SH $\to$ H)**.** Fix integers $m, n \geq 1$ and arity $k \geq 1$. An (m,n)–SuperHyperStructure on S is a map

$$\star : \underbrace{P^m(S) \times \cdots \times P^m(S)}_{k \text{ times}} \longrightarrow P^n(S).$$

Its *hyper–reduction* is the hyperoperation

$$\circ : \underbrace{P(S) \times \cdots \times P(S)}_{k \text{ times}} \longrightarrow P(S), \qquad \circ(A_1, \ldots, A_k) \ := \ \mathrm{Flat}_n\Big(\star\,(\iota_m(A_1), \ldots, \iota_m(A_k)) \Big).$$

Then $\mathcal{H} := (P(S), \circ)$ is a HyperStructure on S.

Definition 14. (Stage 2: de-hyperization H $\to$ Classical)**.** Let $(P(S), \circ)$ be a HyperStructure with $\circ : (P(S))^k \to P(S)$. Choose a strictly order–inducing score $g : S \to \mathbb{Z}$ (or to any totally ordered set) and define the selector

$$\Delta_g : \ P(S) \setminus \{\emptyset\} \longrightarrow S, \qquad \Delta_g(A) := \arg\max_{x \in A} g(x),$$

which is well-defined and single–valued because S is finite and g induces a total order (break ties by a fixed tie–rule if needed). The *classical reduction* is

$$\cdot : \underbrace{S \times \cdots \times S}_{k \text{ times}} \longrightarrow S, \qquad (x_1, \ldots, x_k) \ \mapsto \ \Delta_g(\ \circ\,(\{x_1\}, \ldots, \{x_k\})\ ).$$

Then $\mathcal{C} := (S, \cdot)$ is a classical k-ary structure on S.

Proposition 2. (Recovery on singleton-lifted classical structures)**.** *Suppose a classical k-ary operation $f : S^k \to S$ is* lifted *to an (m,n)–SuperHyperStructure $\star$ by*

$$\star(\iota_m(A_1),\dots,\iota_m(A_k)) \;=\; \iota_n\Big(\{\, f(x_1,\dots,x_k) : x_i \in A_i \,\}\Big),$$

for all $A_1,\dots,A_k \subseteq S$. Then the hyper–reduction (Def. 3.2) satisfies

$$\circ(A_1,\dots,A_k) \;=\; \{\, f(x_1,\dots,x_k) : x_i \in A_i \,\},$$

and the classical reduction (Def. 3.3) applied to singletons recovers f:

$$\cdot(x_1,\dots,x_k) \;=\; \Delta_g(\{f(x_1,\dots,x_k)\}) \;=\; f(x_1,\dots,x_k).$$

Proof: By the nested–singleton identity, $\mathrm{Flat}_n(\iota_n(B)) = B$ for all $B \subseteq S$. Thus

$$\circ(A_1,\dots,A_k) = \mathrm{Flat}_n(\star(\iota_m(A_1),\dots,\iota_m(A_k))) = \mathrm{Flat}_n(\iota_n(\{f(\mathbf{x})\})) = \{f(\mathbf{x})\}.$$

In particular, for singletons $A_i = \{x_i\}$ we obtain $\circ(\{x_1\},\dots,\{x_k\}) = \{f(x_1,\dots,x_k)\}$, hence $\cdot(x_1,\dots,x_k) = \Delta_g(\{f(x_1,\dots,x_k)\}) = f(x_1,\dots,x_k)$. □

Example 11. (Arithmetic on $S = \mathbb{Z}_5$: $(1,2)$–to–Hyper to Classical)**.** Let $S = \mathbb{Z}_5 = \{0,1,2,3,4\}$ and arity $k = 2$. Define an $(m,n) = (1,2)$–SuperHyperStructure

$$\star :\; P(S) \times P(S) \longrightarrow P^2(S), \qquad \star(A,B) := \{\, A \oplus B,\; A \ominus B \,\},$$

where $A \oplus B := \{a + b (\mathrm{mod}\ 5) :\; a \in A, b \in B\}$ and $A \ominus B := \{a - b (\mathrm{mod}\ 5) :\; a \in A, b \in B\}$.

Stage 1 (SH→H). By Definition 3.2 with $m = 1, n = 2$,

$$\circ(A,B) = \mathrm{Flat}_2(\star(A,B)) = (A \oplus B) \;\cup\; (A \ominus B) \;\subseteq S.$$

Stage 2 (H→Classical). Choose $g(x) = x$ (the natural order on $\mathbb{Z}_5$). Then

$$x \cdot y \;:=\; \Delta_g(\circ(\{x\},\{y\})) \;=\; \Delta_g(\{x + y \pmod 5,\; x - y \pmod 5\}).$$

Concrete computation. Take $(x,y) = (1,3)$. Then

$$\{x + y,\; x - y\} \equiv \{\, 1 + 3 = 4,\; 1 - 3 \equiv -2 \equiv 3 \,\} \;\; (\mathrm{mod}\ 5),$$

and $\Delta_g(\{3,4\}) = 4$ since $4 > 3$. Hence $1 \cdot 3 = 4$.

Batch example at the hyper level. Let $A = \{1,4\}$ and $B = \{2,3\}$. Then

$$A \oplus B = \{1+2, 1+3, 4+2, 4+3\} \equiv \{3,4,1,2\} = \{1,2,3,4\},$$

$$A \ominus B = \{1-2, 1-3, 4-2, 4-3\} \equiv \{4,3,2,1\} = \{1,2,3,4\},$$

so $\circ(A,B) = \{1,2,3,4\}$.

Example 12. (SuperHypergraph → Hypergraph → Graph)**.** Let the vertex set be $S = V = \{1,2,3,4\}$. Fix a family of (classical) hyperedges

$$E := \{\{1,2\},\; \{2,3\},\; \{3,4\}\} \;\subseteq P(V).$$

Define an $(m,n) = (1,2)$–SuperHyperStructure of arity $k = 1$ by

$$\star :\; P(V) \longrightarrow P^2(V), \qquad \star(A) := \{\, e \in E \;:\; e \subseteq A \,\}.$$

Thus $\star(A)$ is the family (a subset of E) of all hyperedges fully *realized* inside A.

Stage 1 (SH→H). The hyper–reduction is

$$\circ(A) = \mathrm{Flat}_2(\star(A)) = \bigcup_{e \in E,\; e \subseteq A} e,$$

the set of vertices incident to realized hyperedges in A.

Concrete calculation. For $A = \{1,2,4\}$, the only edge contained in A is $\{1,2\}$, hence $\circ(A) = \{1,2\}$. For $A = \{2,3,4\}$, realized edges are $\{2,3\}$ and $\{3,4\}$, so $\circ(A) = \{2,3,4\}$.

Stage 2 (H→Classical). To obtain a classical *binary* operation on vertices, define

$$x \cdot y \;:=\; \Delta_g(\circ(\{x,y\})), \qquad g(1) < g(2) < g(3) < g(4).$$

If $\{x,y\} \in E$ then $\circ(\{x,y\}) = \{x,y\}$ and $x \cdot y = \max\{x,y\}$; otherwise $\circ(\{x,y\}) = \emptyset$. A total operation results by adopting a fallback (e.g. $x \cdot y := x$) on the empty case. For instance:

$$2 \cdot 3 = \max\{2,3\} = 3 \quad (\text{since } \{2,3\} \in E), \qquad 1 \cdot 4 = 1 \quad (\text{fallback, as } \{1,4\} \notin E).$$

Example 13. (Menus of schedules: $(1,2)$–to–Hyper to Classical with explicit numbers)**.** Let $S = \{\mathrm{M},\mathrm{P},\mathrm{T},\mathrm{A}\}$ denote *Math, Physics, Tower, Aquarium.* Define for $k = 2$ an $(m,n) = (1,2)$–SuperHyperStructure

$$\star :\; P(S) \times P(S) \to P^2(S), \qquad \star(A,B) := \{\, Y \subseteq A \cup B :\; |Y| = 2,\; Y \cap A \neq \emptyset,\; Y \cap B \neq \emptyset \,\}.$$

Thus $\star(A,B)$ is the *family* of all feasible 2-stop day–plans that mix one stop from A and one from B.

Stage 1 (SH→H). Flattening gives the hyperoperation

$$\circ(A,B) = \mathrm{Flat}_2(\star(A,B)) = \bigcup_{Y \in \star(A,B)} Y = A \cup B.$$

Stage 2 (H→Classical). Let $g(\mathrm{M}) = 4,\; g(\mathrm{P}) = 3,\; g(\mathrm{T}) = 2,\; g(\mathrm{A}) = 1$. For singletons $x = \mathrm{T},\; y = \mathrm{P}$,

$$x \cdot y = \Delta_g(\circ(\{x\},\{y\})) = \Delta_g(\{x,y\}) = \Delta_g(\{\mathrm{T},\mathrm{P}\}) = \mathrm{P},$$

and for $x = \mathrm{A},\; y = \mathrm{M}$ we obtain $x \cdot y = \mathrm{M}$. At the hyper level, with $A = \{\mathrm{M},\mathrm{T}\}$ and $B = \{\mathrm{P}\}$ we have $\circ(A,B) = A \cup B = \{\mathrm{M},\mathrm{P},\mathrm{T}\}$.

4|De-Nearization

Intuitively, de-nearization removes conflicting evidence from two descriptively near sets so that the remaining subsets are separated by a prescribed margin ε in the feature space.

Notation 1. *Let (X,Φ) be as in the Near Set definition, with $\Phi(x) \in \mathbb{R}^n$. For $u \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, write*

$$\mathrm{dist}(u,S) := \inf_{v \in S} \|u - v\|_2, \qquad d_\Phi(A,B) := \inf_{u \in \overline{\mathcal{Q}(A)},\, v \in \overline{\mathcal{Q}(B)}} \|u - v\|_2.$$

Then $A\ \delta_\Phi\ B$ iff $d_\Phi(A,B) = 0$. For finite A,B, the closures are unnecessary.

Definition 15. (De-nearization operator via ε–margin filtering)**.** Given $A,B \subseteq X$ and a margin $\varepsilon \geq 0$, define an operator T_ε on pairs by

$$T_\varepsilon(A,B) \;:=\; \Big(\; A', B' \;\Big),$$

where

$$A' = \Big\{ a \in A :\; \mathrm{dist}(\Phi(a),\, \mathcal{Q}(B)) \;\geq\; \varepsilon \Big\}, \qquad B' = \Big\{ b \in B :\; \mathrm{dist}(\Phi(b),\, \mathcal{Q}(A)) \;\geq\; \varepsilon \Big\}.$$

Starting from $(A^{(0)}, B^{(0)}) = (A,B)$, iterate

$$(A^{(t+1)}, B^{(t+1)}) \;:=\; T_\varepsilon(A^{(t)}, B^{(t)}) \qquad (t = 0,1,2,\dots).$$

Any fixed point $(A^\star, B^\star)$ of T_ε reachable by this iteration is called an *ε–de-nearization* of (A,B) and we write

$$\mathrm{DeNear}_\varepsilon(A,B) := (A^\star, B^\star).$$

Remark 2. (Cost–sensitive variant)**.** If removing elements has asymmetric costs $\alpha, \beta \geq 0$ for A and B, one may choose (A', B') solving

$$\min_{A' \subseteq A,\; B' \subseteq B} \alpha\,|A \setminus A'| + \beta\,|B \setminus B'| \quad \text{s.t.} \quad d_\Phi(A', B') \;\geq\; \varepsilon,$$

which always has a solution since the feasible region is finite. The fixed point of T_ε gives a maximal feasible pair; among all such pairs choose a minimum–cost one.

Example 14. (Image patches: removing the conflicting description)**.** Use the probe space from the Near Set example with $\Phi(p) = (\text{mean}, \text{edge}) \in \mathbb{R}^2$. Let

$$\Phi(p_1) = (120, 0.20),\ \Phi(p_2) = (118, 0.21),\ \Phi(p_3) = (100, 0.05),$$
$$\Phi(p_4) = (100, 0.05),\ \Phi(p_5) = (60, 0.04),\ \Phi(p_6) = (200, 0.30),$$

and $A = \{p_1, p_2, p_3\}$, $B = \{p_4, p_5\}$. Take $\varepsilon = 0.01$.

Step $t = 0 \to 1$. Distances from $\Phi(p_3) = (100, 0.05)$ to $\mathcal{Q}(B)$:

$$\min\left\{\|(100, 0.05) - (100, 0.05)\|_2,\ \|(100, 0.05) - (60, 0.04)\|_2\right\} = 0 < 0.01,$$

so p_3 is deleted from A. Similarly for p_4 w.r.t. $\mathcal{Q}(A)$:

$$\min\left\{\|(100, 0.05) - (120, 0.20)\|_2,\ \|(100, 0.05) - (118, 0.21)\|_2,\ 0\right\} = 0 < 0.01,$$

so p_4 is deleted from B. All other points satisfy (explicitly),

$$\|(120, 0.20) - (60, 0.04)\|_2 = \sqrt{(60)^2 + (0.16)^2} > 0.01,$$
$$\|(118, 0.21) - (60, 0.04)\|_2 = \sqrt{(58)^2 + (0.17)^2} > 0.01,$$

and remain.

Fixed point. We obtain $(A^\star, B^\star) = (\{p_1, p_2\}, \{p_5\})$, and

$$d_\Phi(A^\star, B^\star) = \min\left\{\|(120, 0.20) - (60, 0.04)\|_2,\ \|(118, 0.21) - (60, 0.04)\|_2\right\} > 0.01 = \varepsilon.$$

Thus $\text{DeNear}_{0.01}(A, B) = (\{p_1, p_2\}, \{p_5\})$, concretely separating the classes by removing the sole conflicting description $(100, 0.05)$ from both sides.

Example 15. (Devices with price–weight features: cost–sensitive de-nearization)**.** Let $\Phi(x) = (\text{price}/100,\ \text{portability})$ with the data

$$\Phi(\ell_1) = (12, 1),\ \Phi(\ell_2) = (10, 2),$$
$$\Phi(t_2) = (10, 1),\quad \Phi(t_3) = (12, 1).$$

Take $A = \{\ell_1, \ell_2\}$, $B = \{t_2, t_3\}$ and $\varepsilon = 0.5$. Note the conflict: $(12, 1)$ occurs in both A and B so $d_\Phi(A, B) = 0$.

Suppose removing an A–item costs $\alpha = 1$ and removing a B–item costs $\beta = 3$. Compare two feasible candidates:

(i) Drop ℓ_1 *from* A*:* $A' = \{\ell_2\}$, $B' = \{t_2, t_3\}$. Then

$$\text{dist}(\Phi(\ell_2), (10, 1)) = |(2) - (1)| = 1, \quad \text{dist}(\Phi(\ell_2), (12, 1)) = \sqrt{(2)^2 + (1)^2} = \sqrt{5} \approx 2.236,$$

so $d_\Phi(A', B') = \min\{1, \sqrt{5}\} = 1 \geq \varepsilon$. Cost $= 1$.

(ii) Drop t_3 *from* B*:* $A' = \{\ell_1, \ell_2\}$, $B' = \{t_2\}$. Here

$$\min\left\{\|(12, 1) - (10, 1)\|_2,\ \|(10, 2) - (10, 1)\|_2\right\} = \min\{2, 1\} = 1 \geq \varepsilon.$$

Cost $= 3$.

Hence the minimum–cost solution is (i):

$$\text{DeNear}^{\text{cost}}_{0.5}(A, B) = (\{\ell_2\}, \{t_2, t_3\}),$$

with explicit margin $d_\Phi = 1$ and total cost 1.

Theorem 1. (Termination, separation, maximality)**.** *Assume A, B are finite. Then for any $\varepsilon \geq 0$ the sequence $(A^{(t)}, B^{(t)})$ defined above stabilizes in at most $|A| + |B|$ iterations at a fixed point $(A^\star, B^\star)$ satisfying:*

(i) ***(Separation)*** $d_\Phi(A^\star, B^\star) \ \geq\ \varepsilon$.

(ii) ***(Maximality under inclusion)*** *If $A^\star \subseteq \widetilde{A} \subseteq A$ and $B^\star \subseteq \widetilde{B} \subseteq B$ with $d_\Phi(\widetilde{A}, \widetilde{B}) \geq \varepsilon$, then $(\widetilde{A}, \widetilde{B}) = (A^\star, B^\star)$.*

Proof: Each step only deletes points, so $|A^{(t)}|+|B^{(t)}|$ is nonincreasing and strictly decreases whenever a violation occurs; hence stabilization occurs in $\leq |A| + |B|$ steps, yielding a fixed point $(A^\star, B^\star)$. At a fixed point, by construction every $a \in A^\star$ satisfies $\mathrm{dist}(\Phi(a), \mathcal{Q}(B^\star)) \geq \varepsilon$, and every $b \in B^\star$ satisfies $\mathrm{dist}(\Phi(b), \mathcal{Q}(A^\star)) \geq \varepsilon$. Therefore $d_\Phi(A^\star, B^\star) \geq \varepsilon$, proving (i). For (ii), if we add back any deleted element $\hat{a} \in A \setminus A^\star$ (or $\hat{b} \in B \setminus B^\star$), its defining deletion inequality shows $\mathrm{dist}(\Phi(\hat{\ }, \star)) < \varepsilon$ (or the symmetric one), which reduces d_Φ below ε. Hence no proper superset preserves the margin, proving maximality. □

5|Conclusion

In this paper, we introduced three reduction processes: de-roughication, de-hyperization, and de-nearization. De-roughication transforms rough approximations into crisp decision sets using cost-sensitive thresholds. De-hyperization reduces SuperHyperStructures to HyperStructures and then to classical structures through lifting, flattening, and selection procedures. De-nearization separates descriptively near sets by removing conflicting elements while preserving a prescribed feature-space margin. The established properties and illustrative applications demonstrate that these processes provide systematic tools for simplifying uncertain, hierarchical, and higher-order structures. Future work may investigate their extensions to infinite settings, alternative uncertainty models, computational algorithms, and broader practical applications.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Pawlak, Z. (1982). Rough sets. *International journal of computer & information sciences, 11*(5), 341–356. https://doi.org/10.1007/BF01001956

[2] Broumi, S., Smarandache, F., & Dhar, M. (2014). Rough Neutrosophic sets. *Italian journal of pure and applied mathematics, 32*, 493–502. https://doi.org/10.5281/zenodo.30310

[3] Tian, Y., Yan, Y., Zhao, Y., Zhou, P., Zhao, S., & She, Y. (2026). Cost-sensitive fuzzy rough learning for multi-class imbalanced data classification. *International journal of approximate reasoning*, 109784. https://doi.org/10.1016/j.ijar.2026.109784

[4] Yang, S., Qian, J., Yan, S., & Zhi, D. (2026). Dual-scale adaptive fuzzy rough sets. *Machine learning and knowledge engineering for decision making: The 17th international FLINS conference on fuzzy logic for intelligent systems and the 21st international conference on intelligent systems and knowledge engineering (FLINS-ISKE 2026), Sydney, NSW, Australia, Proceedings, Part I* (pp. 416–428). Springer. https://doi.org/10.1007/978-981-92-2369-5

[5] Hsiao, C. C., Chuang, C. C., Jeng, J. T., & Su, S. F. (2013). A weighted fuzzy rough sets based approach for rule extraction. *Proceedings of the 52nd annual conference of the society of instrument and control engineers of Japan (SICE)* (pp. 104–109). IEEE. https://doi.org/10.1109/SICE.2013.6736152

[6] Kumari, N., Gautam, V., & Shreevastava, S. (2026). A review on rough, fuzzy rough and intuitionistic fuzzy rough sets based techniques for attribute reduction. *International journal of system assurance engineering and management*, 1-15. https://doi.org/10.1007/s13198-026-03368-9

[7] Zhang, L., & Zhu, P. (2023). Asymmetric models of intuitionistic fuzzy rough sets and their applications in decision-making. *International journal of machine learning and cybernetics, 14*(10), 3353-3380. https://doi.org/10.1007/s13042-023-01837-2

[8] Chen, R., Liu, Q., Zhang, Y., & Weng, W. (2025). A probabilistic Plithogenic Neutrosophic rough set for uncertainty-aware food safety analysis. *Neutrosophic sets and systems, 90*(1), 59. https://digitalrepository.unm.edu/nss_journal/vol90/iss1/59

[9] Mukherjee, A., Mukherjee, A., Das, R., & Das, A. K. (2026). A new approach of generalized interval-valued Neutrosophic rough set and its application in medical diagnosis. *Transactions on fuzzy sets and systems, 5*(1), 115-134. https://doi.org/10.71602/tfss.2026.1204833

[10] Zhao, H., & Zhang, H. Y. (2020). On hesitant Neutrosophic rough set over two universes and its application. *Artificial intelligence review, 53*(6), 4387-4406. https://doi.org/10.1007/s10462-019-09795-4

[11] Albogami, N. N. (2025). Automated learning style prediction using weighted Neutrosophic fuzzy soft rough sets in E-learning platform. International journal of Neutrosophic science (IJNS), 25(1), 104, https://doi.org/10.54216/IJNS.250109

[12] El-Sheikh, S., Zakaria, A., Said, E., & Adel, A. (2025). Precision in soft rough sets: Covering-based methods enhanced by ideals. *Journal of intelligent & fuzzy systems, 49*(5), 1333-1345. https://doi.org/10.1177/18758967251345852

[13] Fujita, T. (2026). Operations, similarity measures, and rough-set approximations for uncertain sets. *Vietnam journal of science, technology and engineering*. https://doi.org/10.31276/VJSTE.2026.0007

[14] Fujita, T. (2026). Plithogenic fuzzy soft expert set and Plithogenic fuzzy rough set with upside-down logic. *Journal of data science and intelligent systems*. https://doi.org/10.47852/bonviewJDSIS62028065

[15] Fujita, T., & Smarandache, F. (2025). *A dynamic survey of fuzzy, intuitionistic fuzzy, Neutrosophic, Plithogenic, and extensional sets*. Infinite Study. https://fs.unm.edu/ADynamicSurvey.pdf

[16] Fujita, T., & Smarandache, F. (2026). *Handbook of rough set extensions and uncertainty models*. Infinite Study. https://fs.unm.edu/HandbookRoughSetExtensionsUncertainty.pdf

[17] Qiao, S., Zhu, P., & Pedrycz, W. (2022). Rough set analysis of graphs. *Filomat, 36*(10), 3331-3354. https://www.jstor.org/stable/27384815

[18] Pavithra, S., & Manimaran, A. (2023). Uncertainty measure for Z-soft covering based rough graphs with application. *Journal of intelligent & fuzzy systems, 44*(4), 5789-5802. https://doi.org/10.3233/JIFS-223678

[19] Banerjee, M., & Chakraborty, M. K. (2004). Algebras from rough sets. In *Rough-neural computing: Techniques for computing with words* (pp. 157-184). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-18859-6_7

[20] Pawlak, Z. (1998). Rough set theory and its applications to data analysis. *Cybernetics & systems, 29*(7), 661-688. https://doi.org/10.1080/019697298125470

[21] Al-Tahan, M., Davvaz, B., Smarandache, F., & Anis, O. (2021). On some Neutrohyperstructures. *Symmetry, 13*(4), 535. https://doi.org/10.3390/sym13040535

[22] Davvaz, B., Nezhad, A. D., & Nejad, S. M. (2020). Algebraic Hyperstructure of observable elementary particles including the Higgs boson. *Proceedings of the national academy of sciences, India section A: Physical sciences, 90*(1), 169-176. https://doi.org/10.1007/s40010-018-0553-z

[23] Ruggero, M. S., & Vougiouklis, T. (2017). Hyperstructures in Lie-Santilli admissibility and ISO-theories. *Ratio mathematica, 33*, 151. https://doi.org/10.23755/rm.v33i0.374

[24] Smarandache, F. (2024). Foundation of superhyperstructure and Neutrosophic superhyperstructure. *Neutrosophic sets and systems, 63*(2024), 367-381. https://fs.unm.edu/NSS/SuperHyperStructure.pdf

[25] Fujita, T., & Smarandache, F. (2026). *Hyperstructures and Superhyperstructures: Theory and selected applications.* Infinite Study. https://dx.doi.org/10.2139/ssrn.6581458

[26] Kong, X. (2025). Students' innovation ability evaluation in vocational colleges based on the steam education concept: A Neutrosophic superhyper pentapartitioned process model. *Neutrosophic sets and systems, 93*(1), 17. https://digitalrepository.unm.edu/nss_journal/vol93/iss1/17

[27] Fujita, T., & Smarandache, F. (2026). *Representing higher-order networks: A survey of graph-based frameworks.* Neutrosophic Science International Association (NSIA) Publishing House. https://philpapers.org/archive/FUJRHN.pdf

[28] Jech, T. (2003). *Set theory: The third millennium edition, revised and expanded.* Springer-Verlag. https://doi.org/10.1017/S1079898600003358

[29] Smarandache, F. (2022). *Introduction to Superhyperalgebra and Neutrosophic Superhyperalgebra.* Infinite Study. https://books.google.com/books?id=8W9mEAAAQBAJ&printsec=frontcover#v=onepage&q&f=false

[30] Vougioukli, S. (2020). Helix hyperoperation in teaching research. *Science & philosophy, 8*(2), 157-163. http://dx.doi.org/10.23756/sp.v8i2.557

[31] Vougioukli, S. (2020). Hyperoperations defined on sets of S-helix matrices. *Journal of algebraic hyperstructures and logical algebras, 1*(3), 81-90. https://doi.org/10.29252/hatef.jahla.1.3.7

[32] Corsini, P., & Leoreanu, V. (2013). *Applications of hyperstructure theory.* Springer Science & Business Media. https://www.amazon.fr/Applications-Hyperstructure-Theory-Piergiulio-Corsini/dp/1402012225

[33] Novák, M., Křehlík, Š., & Ovaliadis, K. (2019). Elements of Hyperstructure theory in UWSN design and data aggregation. *Symmetry, 11*(6), 734. https://doi.org/10.3390/sym11060734

[34] Oguz, G., & Davvaz, B. (2021). Soft Topological hyperstructure. *Journal of intelligent & fuzzy systems, 40*(5), 8755-8764. https://doi.org/10.3233/JIFS-200242

[35] Smarandache, F. (2023). SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions. *Neutrosophic systems with applications, 12*, 68–76. https://doi.org/10.61356/j.nswa.2023.115

[36] Peters, J. F., & Naimpally, S. A. (2012). Applications of near sets. *Notices of the American mathematical society, 59*(4), 536-542. http://dx.doi.org/10.1090/noti817

[37] Peters, J. F. (2013). Near sets: An introduction. *Mathematics in computer science, 7*(1), 3-9. https://doi.org/10.1007/s11786-013-0149-6

[38] Peters, J. F. (2007). Near sets. Special theory about nearness of objects. *Fundamenta Informaticae, 75*(1-4), 407-433. https://doi.org/10.3233/FUN-2007-751-424

[39] Peters, J. F. (2007). Near sets. General theory about nearness of objects. *Applied mathematical sciences, 1*(53), 2609-2629. https://www.m-hikari.com/ams/ams-password-2007/ams-password53-56-2007/petersAMS53-56-2007.pdf